Semi-Riemannian Symmetric spaces

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1 Symmetric spaces

1.1 Symmetric spaces : definitions and examples

1.1.1 Definition : symmetric space

We begin with a general definition for (real smooth) symmetric spaces.

Definition 1.1.1 *A (differential, real)* **symmetric space** *is a pair* (M, S) *where*

- *M is connected (real smooth) manifold (see comment below)*
- *S is a smooth map $M \times M \longrightarrow M$ satisfying the following Axioms*
  - (SS1) $\quad S(x, x) = x \quad \forall\, x \in M,$
  - (SS2) $\quad S(x, S(x, y)) = y \quad \forall\, x, y \in M,$
  - (SS3) $\quad S(x, S(y, z)) = S(S(x, y), S(x, z)) \quad \forall\, x, y, z \in M,$
  - (SS4) $\quad T_{(x,x)}S(0, \xi) = -\,\xi \quad \forall\, x \in M, \forall\, \xi \in T_xM.$

Comment 1.1.2 Symmetric spaces were first defined in the settings of smooth manifolds. There are now various generalizations. In this paper, we will consider only the classical case and we will not repeat "real smooth". Also manifold will mean real smooth manifold whose topology is Hausdorff and first comptable. Usually, we consider only connected symmetric spaces.

Axioms (SS1) to (SS4) are quite convenient for computations, but their meaning is not so clear. A more "geometric" viewpoint is better for understanding the definition. We first introduce a notation.

Definition 1.1.3 *For all $x \in M$, we* denote *by $s_x : M \longrightarrow M$ the map given by*

$$s_x(y) = S(x, y) \quad \forall\; y \in M,$$

and we call s_x the **symmetry** *(centered) at x.*

Then Axiom (SS1) means that x is a fixed point of s_x and Axiom (SS2) that any symmetry s_x is an involutive diffeomorphism of M (for all x in M). Now Axiom (SS3) may be written as the formula :

$$s_x \circ s_y \circ s_x = s_{s_x(y)}.$$

In words : the conjugate of the symmetry s_y by another symmetry s_x is also a symmetry, more precisely the symmetry at $s_x(y)$. If we consider only Axioms (SS1), (SS2) and (SS3), the corresponding spaces are called "spaces with symmetries" (see Loos).
Now Axiom (SS4) means $T_xs_x = -\,Id_{T_xM}$ (for all $x \in M$). It is often presented through a different (but equivalent) formulation : for any x in M, the point x is an **isolated** fixed point of the symmetry s_x.

1.1.2 Examples of symmetric spaces

- *Example 1* : Take $M = \mathbb{R}^n$ with $S(x,y) = 2x - y$. Then $(\mathbb{R}^n, S)$ is a symmetric space. Here M is the canonical affine space and the symmetry s_x is the (ordinary) affine symmetry through the point x. That example is often refered to as the canonical flat (1-connected) symmetric space.

- *Example 2* : Consider $\mathbb{R}^{n+1}$ with its canonical Euclidean scalar product $\langle .,. \rangle$. Let M be the canonical sphere $\mathrm{S}^n = \{x \in \mathbb{R}^{n+1}; \langle x,x \rangle = 1\}$. We define $S(x,y) = 2\langle x,y \rangle\, x - y$ on S^n, i.e. s_x is the restriction to S^n of the orthogonal symmetry through the line $\mathbb{R}x$. Then (S^n, S) is a symmetric space. That example is often refered to as the canonical Riemannian model space with constant curvature 1.

- *Example 3* : Since we will consider semi-Riemannian symmetric spaces, we give here the following generalization of example 2. We **denote** by $\mathbb{R}^{p,q}$ the affine space $\mathbb{R}^{p+q}$ equipped with the "minkowskian" scalar product $\langle x,y \rangle = \sum_{i=1}^{i=p} x_i y_i - \sum_{j=p+1}^{j=p+q} x_j y_j$. We consider in $R^{p+1,q}$ the "generalized sphere" $\mathrm{S}^{p,q} = \{x \in R^{p+1,q}; \langle x,x \rangle = 1\}$ and we define $S(x,y) = 2\langle x,y \rangle\, x - y$ as above. Then $(\mathrm{S}^{p,q}, S)$ is a symmetric space, refered to as the "canonical" semi-Riemannian model space with signature (p,q) and constant curvature 1.

  Notice that example 2 is the case $q = 0$. Now $\mathrm{S}^{p,q}$ is diffeomorphic to $\mathrm{S}^p \times \mathbb{R}^q$. In particular, $\mathrm{S}^{0,q}$ has 2 connected components and we replace $\mathrm{S}^{0,q}$ by the connected component of the point $(1,0,\ldots,0)$, usually denoted by $\mathrm{S}_0^{0,q}$, which is diffeomorphic to $\mathbb{R}^q$. Similarly, $\mathrm{S}^{1,q}$ is not simply connected, so we replace it by its universal covering, usually denoted by $\widetilde{\mathrm{S}}^{1,q}$. That one is diffeomorphic to $\mathbb{R}^{q+1}$ and it is possible to "lift" the symmetric structure S to $\widetilde{\mathrm{S}}^{1,q}$. For $p \geqslant 2$, the space $\mathrm{S}^{p,q}$ is simply connected. So, with these two replacements for $p = 0$ or 1, we get 1-connected models.

  Here $\mathrm{S}^{p,q}$ is indeed a semi-Riemannian manifold with the structure induced by the restriction of the minkowskian scalar product to its tangent spaces, and its signature is (p,q). If we consider the opposite semi-Riemannian structure, it has signature (q,p). The corresponding semi-Riemannian manifold is denoted by $\mathrm{H}^{q,p}$ and is known as the "canonical" semi-Riemannian model space with signature (p,q) and constant curvature -1, or the "generalized hyperbolic space". Notice that the symmetric structures of $\mathrm{S}^{p,q}$ and $\mathrm{H}^{q,p}$ are obviously the same!

  On the other hand, the canonical Riemannian hyperbolic space $\mathrm{H}^n = \mathrm{H}_0^{n,0}$ has the same symmetric structure as $\mathrm{S}_0^{0,n}$, which is different from S^n.

- *Example 4* : Let L be any connected Lie group. We define a symmetric structure on L by $S(x,y) = xy^{-1}x$. Then (L,S) is a symmetric space. Hence a Lie group is "canonically" a symmetric space. In that way, we get a lot of examples.

Notice that there are symmetric spaces which are not obtained in that way : for example, S^2 is not a Lie group (more precisely, there is no group structure on S^2 whose operations are smooth).

- *Example 5* : The product of two symmetric spaces (M_1, S_1) and (M_2, S_2) is "canonically" a symmetric space for the obvious "product structure" : we take $M = M_1 \times M_2$ and $S((x_1, x_2), (y_1, y_2)) = (S_1(x_1, y_1), S_2(x_2, y_2)), \forall\, x_1, y_1 \in M_1\,, \forall\, x_2, y_2 \in M_2$.

  In a similar way, we may build a symmetric space by the product of any (finite) number of symmetric spaces.

- *Further examples* : We will give some other "classical" examples below in (A COMPLETER), including projective spaces, grassmannians and others.

1.1.3 Definitions : subspace and morphism

Definition 1.1.4 *Let (M, S) be a symmetric space. A* **symmetric subspace** *of M is a connected submanifold N which is "closed" for S, i.e. satisfies : $\forall\, x, y \in N, S(x, y) \in N$.*

Proposition 1.1.5 *A symmetric subspace N of a symmetric space (M, S) is a symmetric space when equipped with the restricition of the symmetric structure S to N.*

Comment 1.1.6 Be carefull that in our Example 2 above, S^n is **not** a symmetric subspace of $\mathbb{R}^{n+1}$, since we modify the symmetric structure. The same remark apply to Example 3. On the other hand, a symmetric space has many subspaces. Here are some examples.

- *Example 1* : Affine subspaces of the canonical affine space $\mathbb{R}^n$ are symmetric subspaces.

- *Example 2* : Any intersection of S^n with a linear subspace in $\mathbb{R}^{n+1}$ is a symmetric subspace (which is also a sphere of lower dimension).

  This is also the case for $S^{p,q}$, at least if the intersection is non void.

- *Example 3* : Any (closed) Lie subgroup of a Lie group is a symmetric subspace.

- *Example 4* : In a product of two symmetric spaces (M_1, S_1) and (M_2, S_2), any "factor" such as $M_1 \times \{x_2\}$ or $\{x_1\} \times M_2$ is a symmetric subspace ($\forall\, x_1 \in M_1\,, \forall\, x_2 \in M_2$).

Definition 1.1.7 *Let (M_1, S_1) and (M_2, S_2) be two symmetric spaces. A* **morphism** *of symmetric spaces (or symmetric morphism for short) from (M_1, S_1) to (M_2, S_2) is a smooth map $f : M_1 \longrightarrow M_2$ such that :*

$$S_2(f(x), f(y)) = f(S_1(x, y)) \quad \forall x, y \in M_1.$$

We may translate the above formula in terms of symmetries :

$$f \circ s_x = s_{f(x)} \circ f \quad \forall x \in M_1.$$

Obviously, symmetric spaces and morphisms form a category, but we will not comment on that here. We only give some elementary examples.

- *Example 1* : Any affine map between canonical affine spaces is a symmetric morphism.
- *Example 2* : The inclusion of a symmetric subspace in a symmetric space is a symmetric morphism.
- *Example 3* : Any smooth group homomorphism between Lie groups is a symmetric morphism.
- *Example 4* : Given a product of two symmetric spaces, the two natural projections onto the two factors are symmetric morphisms.

Once again, notice that the inclusion of S^n in $\mathbb{R}^{n+1}$ is not a symmetric morphism.

1.1.4 Additionnal structures and classifications

Since the category of symmetric spaces contains the category of (connected) Lie group, any "classification" of (all) symmetric spaces or any (general) "structure theorem" has to include a similar result for Lie groups.
To study symmetric spaces, one may procede in the following way : consider classical results for Lie groups and generalize them to symmetric spaces. In this paper, we will consider a different way. Symmetric spaces will be considered as "geometrical objects", and studied for their geometrical properties. As a consequence, we will study classes of symmetric spaces with additionnal (geometric) structures, with a particular attention for semi-Riemannian symmetric spaces. We begin with some definitions.

Definition 1.1.8 *A* **semi-Riemannian symmetric space** (M, S, g) *is*

- *a symmetric space* (M, S)*, together with*
- *a semi-Riemannian structure* g *on* M *such that :*

$$s_x \textit{ is a } g\textit{-isometry } \forall x \in M$$

Comment 1.1.9 If g is a Riemannian structure (i.e. positive definite), then (M, S, g) is called a **Riemannian** symmetric spaces.

Elie Cartan studied Riemannian symmetric spaces starting (A COMPLETER) and classified them. Following Cartan's work, Marcel Berger classified those symmetric spaces which are homogeneous under a simple (or semi-simple) Lie group, which are necessarily semi-Riemannian. But there are many other semi-Riemannian symmetric spaces, contrarily to the Riemannian case.
A classification was given by M. CAHEN and N. WALLACH (A COMPLETER) for the Lorentz case (i.e. signature $(1, n-1)$), and by M. CAHEN and M. PARKER for the case with signature $(2, n-2)$.
We will give a structure theorem for the general case and show how those works took place in the general picture.

- *Example 1* : On the affine space $\mathbb{R}^n$, we consider a minkowskian scalar product, and translate it on $\mathbb{R}^n$, giving a (flat) semi-Riemannian metric g. Then we get a semi-Riemannian symmetric space. In that way, the (unique) canonical symmetric structure on $\mathbb{R}^n$ admits a lot of compatible semi-Riemannian structures.

- *Example 2* : The canonical sphere S^n with its canonical Riemannian structure is a Riemannian symmetric space.

- *Example 3* : In the same way, the space $S^{p,q}$ from (A COMPLETER) is a semi-Riemannian symmetric space, when equipped with the semi-Riemannian structure $g_{p,q}$ induced on its tangent spaces by the minkowskian scalar product of the ambient $\mathbb{R}^{p+1,q}$.
  Also s $H^{p,q}$ is nothing but $S^{q,p}$ with $-g_{q,p}$.

- *Example 4* : Any semi-simple Lie group G has a canonical semi-Riemannian structure, invariant through left and right translations and defined at its unit point e by the Killing form on the Lie algebra of G.

- *Example 5* : Products of semi-Riemannian symmetric spaces are semi-Riemannian symmetric spaces for the product semi-Riemannian structure.

We may consider other structures. Here we only introduce the most classical ones. We will come back to that subject later with the concept of "holonomy" (see A COMPLETER).

Definition 1.1.10 *A symmetric space (M, S) is called*

- **complex** *if M is a complex manifold and s_x is a holomorphic map $\forall x \in M$.*

- **symplectic** *if M admits a symplectic structure ω such that s_x is a symplectic map $\forall x \in M$.*

- **semi-Kähler** *if M admits a semi-Kähler structure (g, J, ω) such that s_x is a automorphism of (g, J, ω) $\forall x \in M$.*

Comment 1.1.11 As above, if g is a Riemannian structure (i.e. positive definite), then a semi-Kählerian symmetric space (M, S, g, J, ω) is called a **Kählerian** symmetric space.
Kählerian symmetric spaces were classified by E. CARTAN among the Riemannian ones. Also, M. BERGER classified semi-Kählerian symmetric spaces among the symmetric spaces which are homogeneous under a simple (or semi-simple) Lie group.
Notice that semi-Kählerian symmetric spaces are symplectic. Ther are symplectic symmetric spaces which are not semi-Kählerian, but they are not "classical". We will give some examples below (see A COMPLETER).
Now we give some examples (and comments) for the complex case.

- *Example 1* : The complex affine space $\mathbb{C}^n$, with $S(x, y) = 2x - y$, is obviously a complex symmetric space. Notice that it is onced again the canonical symmetric space $\mathbb{R}^{2n}$ with a complex structure on it. Also $\mathbb{R}^{2n}$ has a lot of compatible complex structures.

- *Example 2* : We consider on $\mathbb{C}^{n+1}$ the complex-quadratic non-degenerate structure $(z, w) = \sum_{j=1}^{j=n+1} z_j w_j$. The "complex sphere" is $S^n_{\mathbb{C}} = \{z \in \mathbb{C}^{n+1}; (z, z) = 1\}$. We see easily that $S^n_{\mathbb{C}}$ is a complex symmetric space for $S(z, w) = 2(z, w)z - w$.

- *Example 3* : We consider on $\mathbb{C}^{n+1}$ the semi-hermitian non-degenerate structures $\langle z, w\rangle = \sum_{j=1}^{j=p+1} \bar{z}_j w_j - \sum_{k=p+2}^{k=n+1} \bar{z}_k w_k$ and denote $q = n - p$. The subset $S^{2p+1,2q} = \{z \in \mathbb{C}^{n+1}; \langle z, z\rangle = 1\}$ is **not** a complex manifold. [Notive that the notation $S^{2p+1,2q}$ is compatible with the previous one in (A COMPLETER) through an identification between $\mathbb{C}^{n+1}$ and $\mathbb{R}^{2n+2}$].

  Hence we consider the complex projective space $\mathbb{C}P^n$ and the projection $proj : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}P^n$. We denote by $\mathbb{C}P^{p,q}$ the image $proj(S^{2p+1,2q})$. This an open set in $\mathbb{C}P^n$ and it is the whole of $\mathbb{C}P^n$ if $p = 0$ or $q = 0$.

  For any z in $S^{2p+1,2q}$, we consider the hermitian reflexion $s^H_z(w) = 2\langle z, w\rangle z - w$ through the complex vector line $\mathbb{C}z$. Notice that s^H_z is a (complex) bijective linear map with respect to w, so s^H induces a projective isomorphism s_z. Now we may check easily that if $z, w \in S^{2p+1,2q}$ and $[z] = proj(z), [w] = proj(w) \in \mathbb{C}P^n$, then $S([z], [w]) = s_{[z]}([w]) = [s^H_z(w)]$ gives a symmetric strucutre on $\mathbb{C}P^{p,q}$. Moreover, we get a semi-Kählerian structure $g_{p,q}$ on $\mathbb{C}P^{p,q}$, with signature (p, q) [and a Kählerian structure on $\mathbb{C}P^n$ if $q = 0$], through a projection on the tangent spaces to $\mathbb{C}P^n$ at $[z]$ of the scalar product $\langle ., .\rangle$ restricted to the (complex) vector space orthogonal to the line $\mathbb{C}z$.

  Also, we denote by $\mathbb{C}H^{p,q}$ the complex symmetric space $\mathbb{C}P^{q,p}$ together with the semi-Kählerian structure corresponding to $-g_{q,p}$. Here $\mathbb{C}H^{p,0}$ (usually denoted by $\mathbb{C}H^n$) is

Kählerian, but different from $\mathbb{C}P^n = \mathbb{C}H^{n,0}$.

Comment 1.1.12 1. Notice that there exists a construction analoguous to example 3 where we replace $\mathbb{C}$ by the field of quaternions $\mathbb{H}$. The resulting spaces are **not** semi-Káhlerian and not even complex.

2. In example 1 above, $\mathbb{C}^n$ admits (a lot of) semi-Kählerian (and Kählerian) compatible structures.

   On the other hand, the complex sphere $S^n_{\mathbb{C}}$ admits no semi-Kählerian compatible structure, but it admits a compatible (real) semi-Riemannian structure [with signature (n,n)].

3. The "symmetric map" S is a holomorphic map in examples 1 or 2, but not in example 3.

   Hence we may distinguish a special type of complex symmetric spaces, with the following definition.

Definition 1.1.13 *A complex symmetric space* (M, S) *is called* **totally complex** *if the symmetric map* $S : M \times M \longrightarrow M$ *is a holomorphic map.*

So $\mathbb{C}^n$ and $S^n_{\mathbb{C}}$ are totally complex, while $\mathbb{C}P^{p,q}$ and $\mathbb{C}H^{p,q}$ are complex, but not totally complex.

1.2 Canonical groups acting on symmetric spaces

1.2.1 Symmetry group and transvection group

The main point of that section is the following :

Theorem 1.2.1 *A symmetric space is a homogeneous space.*

We will be more precise. Indeed a manifold may be a homogeneous space under the action of a number of different groups. We will give here some "canonical" groups with a transitive action on a given symmetric space.

Definition 1.2.2 *The* **symmetry group** *SG(M,S) (or for short SG) of a symmetric space* (M, S) *is the subgroup of the group of diffeomorphisms of* M *which is generated by the symmetries* s_x *for all* x *in* M*.*
The **transvection group** *TG(M,S) (or for short TG) of a symmetric space* (M, S) *is the subgroup of the group of diffeomorphisms of* M *which is generated by the products of two symmetries* $s_x \circ s_y$ *for all* x *,* y *in* M*.*

There will be two possibilties whether $TG = SG$ or $TG \neq SG$.
For example, if n is even, the symmetry group of the sphere S^n is the special orthogonal group $SO(n+1)$ and $TG = SG$. On the other hand, if n is odd, the symmetry group of the sphere S^n is the full orthogonal group $O(n+1)$ while the transvection group is $SO(n+1)$, so $TG \neq SG$ here.
In the case $TG \neq SG$, then TG is a normal subgroup of SG with index 2. All the symmetries s_x belong to $SG - TG$ and SG is a semi-direct product of TG with the subgroup of order 2 generated by one symmetry s_a for a choosen point a in M.
As an example, the tranvection group TG of the canonical symmetric space $\mathbb{R}^n$ is its translation group (also denoted $\mathbb{R}^n$) and its symmetry group SG is such that $SG - TG$ is exactly the set of all symmetries s_x for any x in $\mathbb{R}^n$.

Proof of the Theorem : We prove that TG acts transitively on M, so M is a homogeneous space under the action of both TG and SG.
(A REVOIR) Given any points x, y in M, we consider the map $T_{x,y} : M \longrightarrow M$ defined by $T_{x,y}(z) = S(y, S(x, z)) = s_y \circ s_x(z)$. So $T_{x,y}$ is in TG. If $x = y$, then $T_{x,x}$ is the identity of M

1.2.2 Automorphism group

We introduce another "canonical" group acting transitively on M.

Definition 1.2.3 *The* **automorphism group** *AG(M,S) (or for short AG) of a symmetric space (M, S) is the group of symmetric automorphisms of (M, S).*

We recall that $\alpha : M \longrightarrow M$ is a symmetric automorphism of (M, S) if it is a diffeomorphism of M and satisfies : $S(\alpha(x), \alpha(y)) = \alpha(S(x, y)) \quad \forall x, y \in M$.
Now the above formula may be written : $s_{\alpha(x)} \circ \alpha = \alpha \circ s_x$ or equivalently $\alpha \circ s_x \circ \alpha^{-1} = s_{\alpha(x)}$.
This last formula may be translated in words as "the conjugaison by α of the symmetry at x is the symmetry at $\alpha(x)$".
The axiom (SS3) for symmetries shows that any symmetry s_x is a symmetric automorphism, so SG is incuded in AG. As a consequence, AG acts transitively on M.
The automorphism group of the canonical symmetric space $\mathbb{R}^n$ is the group of affine transformations, so it is much lager than its symmetry group. On the other hand, the automorphism group of the sphere S^n is $O(n+1)$. So $AG = SG$ for an odd dimensional sphere.

In the case where the symmetric space (M, S) is given with some compatible structure as in 1.1.4 above, we may consider other groups related to the struture.
First we have the group of automorphisms of the given structure :

- isometry group for a semi-Riemannian or semi-Kählerian structures;

- symplectic group for a symplectic structure;
- holomorphic group for a complex structure.

And we may consider the group of automorphisms of the structure which are also automorphisms of the symmetric structure. We will denote this group by IG.
Notice that isometries of semi-Riemannian (or semi-Kählerian) structures are always automorphisms of the symmetric structure. But that property is no more true for the symplectic or the complex case. This is already the case for $\mathbb{R}^{2n}$ with its canonical symplectic structure, and for $\mathbb{C}^n$ with its canonical complex structure.
Notice that with our definition of a compatible structure any symmetry s_x is in IG, so we have $TG \ \subset \ SG \subset \ IG \ \subset \ AG.$.
We will show below that

- AG is (in a canonical way) a Lie group which acts smoothly on M,
- the subgroups TG, SG and IG are Lie subgroups,
- TG and SG are normal subgroups of IG and AG
- TG is connected.

1.2.3 Isotropy groups

Let (M, S) be a symmetric space and G be one of the groups TG, SG,IG or AG. More generally we may choose any closed Lie subgroup of AG which contains TG, but we will suppose moreover that the subgroup G is invariant in AG through conjugation by (at least) one symmetry s_x. That is obviously the case for SG, IG or AG, but it is also true for TG..

Definition 1.2.4 *We choose a base point $a \in M$. The* **isotropy subgroup** *of G at a, denoted H_a (or for short H) is the closed subgroup*

$$H = \{\alpha \in G; \alpha(a) = a\}.$$

Now the canonical map (associated to a) $ev_a : G \longrightarrow M$ given by $ev_a(\alpha) = \alpha(a)$ induces a canonical identification of M with the quotient G/H.

Definition 1.2.5 *We denote by σ the involutive automorphism of G induced by the conjugation by the symmetry s_a, i.e. $\sigma(\alpha) = s_a \circ \alpha \circ s_a \quad \forall \alpha \in G$.*
We denote by G^σ the closed subgroup of G given by the fixed points of σ, and by G_0^σ the connected component of the identity in G^σ.

We see easily that G^σ is exactly the subgroup of elements in G which commutes (in AG) with the symmetry s_a.

Proposition 1.2.6 *We have $G_0^\sigma \subset H \subset G^\sigma$.*

Example : We consider the canonical sphere S^n and choose $a = (1, 0, \cdots, 0)$. Then $AG = O(n+1)$ and $AG^\sigma = O(1) \times O(n)$ has 4 connected components, with $AG^\sigma = \{1\} \times SO(n)$. Now $H = \{1\} \times O(n)$.
In the whole theory of symmetric space, we will not consider that isotropy subgroup as an abstract Lie group, but we will always consider it together with its canonical isotropy representation.

Definition 1.2.7 *We denote by V the vector space T_aM tangent to M at a. The* **isotropy representation** *ρ of M at a is the linear representation of the isotropy subgroup H at a given by*

$$\rho(\alpha) = T_a\alpha \quad \in \quad Gl(V).$$

Usually, we will consider it for the case of the transvection group TG.
Examples :

- For the canonical affine space $\mathbb{R}^n$, we have $TG = \mathbb{R}^n$, $H = \{Id\}$, $V = \mathbb{R}^n$ and ρ is the trivial representation on $\mathbb{R}^n$.
- For the canonical sphere S^n, we have $TG = SO(n+1)$, $H = SO(n)$, $V = \mathbb{R}^n$ and ρ is (isomorphic to) the canonical representation of $SO(n)$ in $\mathbb{R}^n$.
- For a (connected) Lie group L, viewed as a symmetric space, we may show that TG is the "adjoint" group, i.e. the quotient L/Z of L by its center Z, and ρ is the adjoint representation of L/Z on the Lie algebra of L.

Remarks : (1) Conversely, we may consider a pair (G, σ), where G is a (connected) Lie group and σ an involutive automorphism of G. then we take any subgroup H with $G_0^\sigma \subset H \subset G^\sigma$. Then we may show that the quotient G/H becomes a symmetric space.
Some authors even define symmetric spaces in that way. Notice that if G is a simple group, then G is indeed the corresponding transvection group. In the general case, we may get the same symmetric space with various groups G. That is the main reason for our choice of the "classical" definition.

1.2.4 More classical examples and coverings

With the preceding section, we may get many "classical" symmetric spaces through the involutive automorphisms of the "classical" groups. We describe here some of them from a "geometric" viewpoint.
Example 1 : We consider the vector space $E = \mathbb{R}^n$ and the set $D(p, n-p; \mathbb{R})$ of all decomposition of E in a direct sum $F_1 \bigoplus F_2$ where F_1 is a subspace with dimension p. Notice that such

a direct sum is characterized by an involutive automorphism f of E with $Ker(f - Id) = F_1$ and $Ker(f + Id) = F_2$. Then $D(p, n - p; \mathbb{R})$ is a symmetric space in the following way : we consider $Gl^+(E)$ and the involutive automorphism given by the conjugation with one of these f.

Example 2 : We may also take $\mathbb{R}^n$ with a non-degenerate symmetric or symplectic 2-form, and consider the orthogonal direct sums with two non-degenerate subspaces. Here the involutive automorphism$f$ is an isometry.

Example 3 : In the preceding examples, we may replace the field $\mathbb{R}$ by the fields $\mathbb{C}$ or $\mathbb{H}$. And we may take non-degenerate (complex-)symmetric, (complex-)symplectic of (semi-)hermitian forms on $\mathbb{C}^n$, and (semi-)hermitian or skew-hermitian forms on $\mathbb{H}^n$.

Example 4 : On $\mathbb{R}^{2n}$ with a symplectic form or a symmetric form with signature (n, n) we may also consider direct sums of two totally isotropic subspaces (with dimension n). We may also consider $\mathbb{C}^{2n}$ with a (complex-)symplectic from, a non-degenerate (complex-)symmetric form or a semi-hermitian form with signature (n, n). And we may consider $\mathbb{H}^{2n}$ with a non-degenerate skew-hermitian form or a semi-hermitian form with signature (n, n).

Example 5 : Inside $Gl(n, \mathbb{R})$, we may consider the subset of symmetric matrices, which contains different connected components according to the signature $(p, n - p)$. Then each connected component is a symmetric subspace of the group $Gl(n, \mathbb{R})$ with its symmetric structure.

Example 6 : Notice that we get a **different** symmetric space by considering only $Sl(n, \mathbb{R})$, i.e. symmetric matrices with determinant 1 and a given signature $(p, n - p)$.

Example 7 : In the preceding examples 5 and 6, we may consider skew-symmetric (or alternate) matrices instead of symmetric ones. We may also replace $\mathbb{R}$ by $\mathbb{C}$ or $\mathbb{H}$, and consider hermitian or skew-hermitian matrices.

Coverings : Any connected covering of a symmetric space is still a symmetric space. Especially, the universal covering of a symmetric space is a symmetric space.

On the other hand, if the universal covering of a manifold is a symmetric space, the manifold itself is not necessarily a symmetric space, but the local structure is the same as a symmetric space.

1.3 Canonical connection

1.3.1 The canonical connection of a symmetric space

A symmetric space is in a canonical way an "affine" space, i.e. it admits a canonical (torsion-free) connection with many properties. Be carefull that it is not necessarily a "flat" affine space, since the connection may have non null curvature. We begin with the precise statement :

Theorem 1.3.1 *Let (M, S) be a symmetric space. Then there exists on M a unique connection ∇ which is invariant under the symmetries s_x for all x in M.*
Moreover ∇ is torsion free and its curvature R satisfies $\nabla R = 0$.

Definition 1.3.2 *We call ∇ the* **canonical connection** *of the symmetric space (M, S).*

We recall that $T_x s_x = -Id_{T_xM}$. Hence, if a connection ∇ is invariant by s_x, then any tensor built from ∇ with an odd number of components is necessarily 0 at x. That shows that the torsion T^∇ and the derivative of the curvature ∇R^∇ are null. On the other hand, the curvature is a $(1, 3)$ tensor, so it is not necessarily null.
The existence of the canonical connection is not so obvious (A COMPLETER).
For further use, we recall some properties of the curvature :

Proposition 1.3.3 *Let (M, S) be a symmetric space, and ∇ its canonical connection. Then the curvature R of ∇ satisfies the following 3 properties, $\forall\ V, W, X, Y, Z \in T_xM$ and $\forall\ x \in M$:*
(R1) The curvature is alternative :

$$R(X, Y)Z = -R(Y, X)Z;$$

(R2) "First Bianchi identity" :

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0;$$

(R3) "Ricci identity" :

$$R(V, W)R(X, Y)Z = R(R(V, W)X, Y)Z + R(X, R(V, W)Y)Z + R(X, Y)R(V, W)Z.$$

Moreover, for any x in M and any α in the isotropy group AH at x (with respect to the automorphism group AG), the curvature R at x is invariant by $\rho(\alpha)$.

Notice that :

- the first property is valid for the curvature of any connection;
- the first Bianchi identity is valid for any torsion-free connection;
- there exists a "second" Bianchi identity on ∇R^∇, but it is void here;
- in the torsion-free case, the Ricci identity involves also the second derivative $\nabla\nabla R^\nabla$, which is null here.

The last assertion of the theorem follows from :

Theorem 1.3.4 *The automorphism group AG of a symmetric space coincides with the affine group of the canonical connection ∇ [i.e. the group of diffeomorphisms of M which preserves the connection].*
Hence the curvature tensor is invariant by any element in AG.

1.3.2 Geodesics

Since the symmetries s_x preserve the connection ∇, they also preserve all objects associated to the connection, in particular the geodesics and the parallel transport along any smooth curve.
First we show that the geodesics are exactly the one-dimensional symmetric subspaces of the given symmetric space.

Proposition 1.3.5 *Let (M, S) be a symmetric space.*

1. *The canonical connection ∇ is geodesically complete.*
2. *For any geodesic $c : \mathbb{R} \longrightarrow M$, we have :*
$$s_{c(u)}(c(t)) = c(2u - t).$$
3. *The geodesics of ∇ are exactly the non-constant symmetric homomorphisms from $\mathbb{R}$ to M.*

Now we relate the transvections to the parallel transport.

Proposition 1.3.6 *Let (M, S) be a symmetric space.*

1. *For any geodesic $c : \mathbb{R} \longrightarrow M$, we have :*
$$s_{c(v)} \circ s_{c(u)}(c(t)) = c(t + 2(v - u)).$$
2. *The transvection $tr_{c,w} = s_{c(t+w)} \circ s_{c(t)}$ depends only on the geodesic c and w, and not on t.*
3. *We have : $tr_{c,t} \circ tr_{c,u} = tr_{c,t+u}$ and $tr_{c,-t} = (tr_{c,t})^{-1}$.*
4. *The tangent map $T_{c(t)}tr_{c,w}$ of the transvection $tr_{c,w}$ at the point $c(t)$ of the geodesic c coincides with the parallel transport along c from $c(t)$ to $c(t + 2w)$.*

Then we may deduce :

Corollary 1.3.7 *A geodesic in a symmetric space (M, S) is either injective or periodic.*
If a geodesic is periodic with period w, it is a symmetric homomoprhism of a "circle" $S^1_w = \mathbb{R}/_{w\mathbb{R}}$ into M.

Now that has consequences for any symmetric subspace.

Corollary 1.3.8 *A symmetric subspace of a symmetric space (M, S) is totally geodesic.*

1.3.3 Holonomy

Once again, we choose a base point a in the symmetric space (M, S), and we call V the tangent vector space to M at a, i.e. $V = T_a M$.
We recall that the holonomy group H^∇ of the connection ∇ at the point a is the subgroup of the linear group $Gl(V)$ generated by all the parallel transports along the closed piece-wise smooth curves in M with origin and end in a.
The restricted holonomy subgroup H^∇ is the connected component of the identity in H^∇. It also the subgroup of H^∇ generated by the parallel transport along those closed piece-wise smooth curves in M [with origin and end in a] which are homotopic to the constant curve in a. It is known that H^∇ is a Lie subgroup of $Gl(V)$, but it is not necessarily closed.
We consider now the Lie subalgebra $\mathcal{H}^\nabla$ of H_0^∇ in the Lie algebra $\mathcal{GL}(V)$ [which is also the Lie subalgebra corresponding to H^∇]. We call it the holonomy subalgebra of (M, S) at a.
For a symmetric space, the (restricted) holonomy is related to the curvature tensor in the following way :

Theorem 1.3.9 *Let (M, S) be a symmetric space and a a base point in M.*
The holonomy subalgebra of (M, S) at a is exactly the linear subspace of $\mathcal{GL}(V)$ generated by all the endomorphisms $R(X, Y)$ for all vectors X, Y in $V = T_a M$.

With the relation between parallel transport and transvection we have :

Theorem 1.3.10 *Let (M, S) be a symmetric space and a a base point in M.*
The holonomy subalgebra of (M, S) at a is exactly the Lie subalgebra of the isotropy subgroup TH at a for the action of the transvection group TG on M.

1.3.4 Lie or symmetric triple system

We begin with a definition which will summarize most properties of the curvature at a point in a symmetric space.

Definition 1.3.11 *Let V be a vector space. A* **Lie triple system** *(or symmetric triple) is a 3-linear map : $[.,.,.] : V \times V \times V \longrightarrow V$ with the following properties for any X, Y, Z, U, W in V:*

- *(LT 1) it is alternate for thefirst two variables, i.e.*

$$[X, Y, Z] = -[Y, X, Z] \ ;$$

- *(LT 2) it satisfies a Jacobi identity :*

$$[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0 \ ;$$

- *(LT 3) it satisfies :*

$$[U,W,[X,Y,Z]] = [[U,W,X],Y,Z] + [X,[U,W,Y],Z] + [X,Y,[U,W,Z]] \ .$$

We have immediately :

Proposition 1.3.12 *Let (M,S) be a symmetric space, a a base point in M, and $V = T_aM$. Then the curvature tensor at a : $(X,Y,Z) \longrightarrow R(X,Y)Z$ is a Lie triple system.*

Notice that we may build what may seem other examples :
Example : Let $\mathcal{L}$ be any Lie algebra. Then the map : $(X,Y,Z) \longrightarrow [[X,Y],Z]$ is a Lie triple system.
But those are also symmetric examples with a suitable identification of the tangent space at the unit element of a connected Lie group L with its Lie algebra $\mathcal{L}$. Then L has a symmetric structure and the curvature of the canonical (symmetric) connection is $\frac{1}{4}[[X,Y],Z]$.
Now we consider a Lie triple system $(V,[.,.,.])$ and build a symmetric space with it.
We consider the linear subspace $\mathcal{H}$ of $\mathcal{G}\updownarrow(V)$ generated by all the map $ev_{X,Y} : Z \longrightarrow [X,Y,Z]$ for all the vectors X,Y in V. With (LT 3), we see that $\mathcal{H}$ is a subalgebra of $\mathcal{G}\updownarrow(V)$.
Then we consider the vector space $\mathcal{G}$ which is the direct sum $\mathcal{H}\bigoplus V$. We define a bracket $[.,.]$ on $\mathcal{G}$ in the following way :

- if $X,Y \in V$, then $[X,Y] = ev_{X,Y} \ \in \ \mathcal{G}$,
- if $A \in \mathcal{G}$ and $X \in V$, then $[A,X] = A(X) \in V$,
- if $A,B \in \mathcal{G}$, then $[A,B]$ is the bracket in $\mathcal{H}$.

Proposition 1.3.13 *The bracket $[.,.]$ defined above is a Lie algebra structure on $\mathcal{G}$.*

Define the linear map $\sigma : \mathcal{G} \longrightarrow \mathcal{G}$ by $\sigma(A,X) = (A,-X)$ for any $A \in \mathcal{H}$ and $X \in V$.
Then σ is an involutive automorphism of the Lie algebra $\mathcal{G}$.
The "first factor" $\mathcal{H} = \mathcal{H} \times \{0\}$ is the Lie subalgebra of $\mathcal{G}$ formed with the vectors fixed under σ.
We denote by $\mathcal{Q}$ the "second factor" $\{0\} \times V$. Then we have :

$$[\mathcal{H},\mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H},\mathcal{Q}] \subset \mathcal{Q}, \quad [\mathcal{Q},\mathcal{Q}] = \mathcal{H}.$$

Now let $\widetilde{G}$ be the connected and simply connected Lie group with Lie algebra $\mathcal{G}$, The automorphism σ of the Lie algebra $\mathcal{G}$ induces a Lie group involutive automorphism (still denoted by σ) on $\widetilde{G}$. We denote by $\widetilde{H}^\sigma$ the closed subgroup formed by the fixed points under σ. Now let $Z(\widetilde{G})$ be the center of $\widetilde{G}$ and $Z = Z(\widetilde{G})\bigcap \widetilde{H}^\sigma$. Since Z is a closed normal subgroup of $\widetilde{H}^\sigma$ and $\widetilde{G}$, we may define the quotients : $G = \widetilde{G}/Z$ and $H^\sigma = \widetilde{H}^\sigma/Z$. Furthermore, Z is

(point-wise) σ-invariant and σ induces a involutive automorphism (still denoted by σ) on G, whose fixed points form H^σ.
Then the quotient $M = G/H^\sigma$ is a symmetric space whose transvection group is precisely G.
One may also take any subgroup H such that $H_0^\sigma \subset H \subset H^\sigma$, where H_0^σ is as always the connected component of the unit element in H^σ
The corresponding quotients G/H are also symmetric spaces, which are coverings of the preceding ones.
With $H = H_0^\sigma$, the quotient G/H_0^σ is even simply connected.

1.3.5 Final Remarks

In the preceding two chapters, we saw that there are many ways to consider symmetrci spaces.

- In A REVOIR { sssec111} , we gave a general "algebraic" definition, where a symmetric space is defined through an "operation", i.e. a map with "algebraic" axioms.
- In A REVOIR { sssec123} , we presented symmetric spaces as special homogeneous spaces
- In A REVOIR { sssec134} , we introduced Lie triple systems, which correspond to the "local" description of a symmetric space near some base point

In the following, we will focuss on the infinitesimal viewpoint. In particular, we will consider the following objects :

- the holonomy representation
- the adjoint representation of the transvection group
- the curvature tensor (at some base point)

Indeed, the infinitesimal approch describes symmetric spaces up to covering.

2 Representations and invariant subspaces

2.1 Introduction and definitions

2.1.1 Introduction

In this chapter, we will consider "ordinary" representations : i.e. finite dimensional linear representations of Lie groups, but also representations of Lie algebras. We will be interested in the invariant subspaces. The general procedure will be exactly the same for both cases.
So we will consider a general setting which will cover all the cases.

Definition 2.1.1 *A (general) representation is a (non void) subset M of the space of endomorphisms $End(E)$ of a finite-dimensional vector space E.*

For the case of a representation ρ of a Lie group G or a Lie algebra $\mathcal{G}$ we take $M = \rho(G)$ or $M = \rho(\mathcal{G})$.
Also we may consider as well real, complex or quaternionic vector spaces. The basic technics are completely algebraic in this chapter, so they apply to any field. In order to have both the real and the complex case, we will denote by $\mathbb{K}$ the field.
In this chapter, a representation will be a general representation in the above sense.

Definition 2.1.2 *Given a representation $M \subset End(E)$ of a finite-dimensional vector space E, a linear subspace F is called M-invariant (or for short invariant) if it is invariant by any element f in M.*

Notice that the intersection or the sum of two invariant subspaces are obviously invariant subspace. Hence the set of all invariant subspaces of a representation is a lattice (a sub-lattice of the lattice of all linear subspaces of E).

Definition 2.1.3 *Given a representation $M \subset End(E)$ of a finite-dimensional vector space E, we denote by $\mathcal{L}(\mathcal{M})$ the lattice of M-invariant subspaces of E.*

In this chapter, we will try to describe such a lattice. More precisely, we will not try to enumerate all the elements, but instead look for "canonical" elements in $\mathcal{L}(\mathcal{M})$ and also for "canonical" maximal flags maid with elements of $\mathcal{L}(\mathcal{M})$.

Definition 2.1.4 *Given a representation $M \subset End(E)$ of a finite-dimensional vector space E, a maximal flag in $\mathcal{L}(\mathcal{M})$ is an ordered finite family of M-invariant subspaces :*

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{m-1} \subset F_m \subset F_{m+1} = E$$

such that for all $j = 0, \cdots, m \quad F_j \neq F_{j+1}$ and there is no invariant subspace $F \neq F_j$ orF_{j+1} such that $F_j \subset F \subset F_{j+1}$.

We precise now what we consider as "canonical". In the lattice $\mathcal{L}(\mathcal{M})$, there may be an infinite number of subspaces and even continuous families. We will be interested by some subspaces which are "unique" (for some property). We will consider some of them that we get through some precise construction. They will share a common "general" property, that is to be invariant under any automorphism of the full picture.

Definition 2.1.5 *Given a representation $M \subset End(E)$ of a finite-dimensional vector space E, an automorphism of the representation is a linear automorphism θ of E such that $\theta \circ f \in M, \forall f \in M$.*
Such a θ is called a commuting automorphism of moreover $\theta \circ f = f, \forall f \in M$.

The set $Aut(M)$ of automorphisms of the representation M is a closed subgroup of $Gl(E)$, and the set $Com(M)$ of commuting automorphisms is a closed normal subgroup of $Aut(M)$. So we will be interested by some of the subspaces in $\mathcal{L}(\mathcal{M})$ which are invariant by any element of $Aut(M)$. Notice that we do not pretend to give here all the $Aut(M)$-invariant subspaces.

2.1.2 Simple and semi-simple representations

Given a representation $M \subset End(E)$ and a subspace $F \in \mathcal{L}(\mathcal{M})$, we may retrict the action of the elements of M to the subspace F. We get in that way another representation $M^F \subset End(F)$, where M_F is the set oif restrictions of the elements of M to F.
We call $M_F \subset End(F)$ a **sub-representation** of M. Since it is characterized by the invariant subspace F, we will say for short that "F is a subrepresentation".
We may also consider the quotient space E/F and the set $M_{G/F}$ of endomorphisms of G/F induced by the elements of M. We get another representation $M_{G/F} \subset End(G/F)$.
We call $M_{G/F} \subset End(G/F)$ a **quotient representation** of M. As above, we will say for short that "G/F is a quotient representation.

Definition 2.1.6 *A representation $M \subset End(E)$ is called* **simple** *if $E \neq \{0\}$ and there exists no non-trivial invariant subspace, i.e. $\mathcal{L}(\mathcal{M})$ contains only the two "trivial" elements 0 and E.*
A representation $M \subset End(E)$ is called **semi-simple** *if it is a direct sum of simple representations.*

Notice that a 1-dimensional representation is always simple in the above sense. Some authors exculde that 1-dimensional case of the definition of "simple" representations, but here we will not exclude it.

Proposition 2.1.7 *A representation is semi-simple if and only if $\forall F \in \mathcal{L}(\mathcal{M})$, there exists (at least one) $F' \in \mathcal{L}(\mathcal{M})$ such that $E = F \bigoplus F'$.*

Notice that for a semi-simple representation, its decomposition in a (finite) direct sum is not unique. We may only show that the simple factors are unique up to order and isomorphisms. An illustration of that fact is the following "trivial" case.
We recall that all the linear suspaces of E are invariant in a representation M if and only if M is a subset of the space of homotheties of E, i.e. $M \subset Ht(E) = \{\lambda Id_E \in Gl(E); \forall \lambda \in \mathbb{K}\}$. Such a "trivial" representation is indeed semi-simple and we may choose any decompositions of E in a direct sum of 1-dimensional subspaces.

2.1.3 The "socle" of a representation

Given a representation $M \subset End(E)$, we consider all the **simple** subrepresentations F. We see easily that there exists at least one simple subrepresentation in any representation : we take one invariant suspace F with minimal dimension (among the invariant subspaces).
The subspace F for a simple subrepresentation is not necessarily "canonical" (in our sense), so we consider all of them altogether :

Definition 2.1.8 *The* **socle** *$S(M, E)$ (or for short $S(E)$ or S_1) of a representation $M \subset End(E)$ is the sum of all subspaces of simple subrepresentations of $M \subset End(E)$.*

Proposition 2.1.9 *The socle $S(E)$ of a representation $M \subset End(E)$ is an (non-zero) invariant subspace.*
The subrepresentation $S(E)$ is semi-simple.
For any semi-simple subrepresentation of M, the corresponding subspace F is included in $S(E)$.
The socle $S(E)$ coincides with E if and only if M is a semi-simple representation.

In other words, $S(E)$ is the invariant subspace of the "maximal" semi-simple subrepresentation of M, i.e. it contains all the other subspaces of semi-simple subrepresentations.
We see easily that $S(E)$ is "canonical" : it is given by a precise construction, and it is obviously invariant by any automorphism of the given representation.

2.1.4 Duality

We will be interested in particular by minkowskian representations. In that case the orthogonal subspace of any invariant subspace of a representation is also invariant if M "preserves" the scalar product, or at least preserves the orthogonality relation between subspaces.
[Be carefull that "preserving the scalar product" has a different meaning for representations of Lie groups or representations of Lie algebras, but "preserving the orthogonality relation" is the same for a connected Lie subgroup of $Gl(E)$ and its associated Lie algebra.]
Also a non-degenerate scalar product gives a linear isomorphism between E and its "dual" space $E^\star = Hom(E, \mathbb{K})$. In that case the socle of a representation coincides with the socle of the "contragrediant" representation in the following sense :
If $M \subset Gl(E)$ is a representation, define $M^\star = \{f^\star \in End(E^\star)\ ; \forall f \in M\}$. Then $M^\star \subset End(E^\star)$ is also a representation, called the **contragrediant** representation of $M \subset Gl(E)$.
We introduce below a definition which will be interesting both for the orthogonal of the socle in the minkowskian case, and for the duality in the general case.

Definition 2.1.10 *A subrepresentation F of a representation $M \subset End(E)$ is called* **co-simple** *if the associated quotient representation $M_{G/F} \subset End(G/F)$ is simple.*

A subrepresentation F of a representation $M \subset End(E)$ is called **co-semi-simple** *if the associated quotient representation $M_{G/F} \subset End(G/F)$ is semi-simple, or equivalently if the subspace F is the intersection of the subspaces of co-simple representations.*

Notice that a F is co-simple (respectively co-semi-simple) for a representation M if and only if its orthogonal $F^{\perp}$ si simple (respectively semi-simple) for the contragrediant representation $M^{\star}$.
If M leaves invariant the orthogonality relation for subspaces (with respect to a non degenerate scalar product or a non-degenrate symplectic form), then the orthogonal $F^{\perp}$ in the dual space is sent to the usual orthogonal of F in E (so we use the same notation for them).

2.1.5 Minimal radical

Once again, there exists at least one co-simple representation if M is not a simple representation itself. And we may consider all the co-simple representations.

Definition 2.1.11 *The* **minimal radical** $R(M, E)$ *(or for short $R(E)$ or R_1) of a representation $M \subset End(E)$ is the intersection of all subspaces of co-simple subrepresentations of $M \subset End(E)$.*

We may have called it the "co-socle" but it is indeed related to various notions of "radical".

Proposition 2.1.12 *The minimal radical $R(E)$ of a representation $M \subset End(E)$ is an invariant subspace, always different from E.*
The subrepresentation $R(E)$ is semi-simple or $R(E) = \{0\}$
For any co-semi-simple subrepresentation of M, the corresponding subspace F contains $R(E)$.
The socle $R(E)$ is $\{0\}$ if and only if M is a semi-simple representation.

In other words, $R(E)$ is the invariant subspace of the "minimal" co-semi-simple subrepresentation of M, i.e. it is included in all the other subspaces of co-semi-simple subrepresentations. We see easily that $R(E)$ is "canonical" : it is given by a precise construction, and it is obviously invariant by any automorphism of the given representation.

2.2 Canonical Jordan-Hölder decomposition of a representation

2.2.1 Jordan-Hölder decompositions

"Jordan-Hölder decomposition" is a very general notion, which is used in many settings. Here we will adapt it to our notion of representation.

Definition 2.2.1 *Given a representation $M \subset End(E)$, a Jordan-Hölder decomposition of M is a family*

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{m-1} \subset F_m \subset F_{m+1} = E$$

of invariant subspaces such that :

$$\forall j = 0, \cdots, m \quad \text{the quotient representations } F_{j+1}/F_j \text{ are semi} - \text{simple.}$$

We will **not** consider the general theory of Jordan-Hölder decompositions. We only recall that we may refine such a decomposition such that all quotients are simple, and then the quotients are unique up to order and isomorphisms.
Once again, we will be interested by "canonical" Jordan-Hölder decompositions. It is easy to get two of them by using in an iterative way either the socle or the radical.
But here we want to use both the socle and the radical altogether. Since the following construction seems to be new (at least for the author), we will give details below. In particular, we will detail the various steps of the iteration in the decompositions that we will consider.
We will introduce some more invariant subspaces through iteration.

2.2.2 First step

Let $M \subset End(E)$ be a representation. Then the socle $S_1 = S(E)$ and the minimal radical $R_1 = R(E)$ are invariant subspaces, the subrepresentation S_1 is semi-simple and the quotient representation E/R_1 is also semi-simple.
We consider the intersection $S_1 \cap R_1$ and the sum $S_1 + R_1$.
Our first step will be the (complete) sublattice $L_1(E)$ of $\mathcal{L}(\mathcal{M})$ given by :

$$\{0\},\ R_1 \cap S_1,\ R_1,\ S_1,\ R_1 + S_1,\ E\ .$$

We may extract two "maximal flags" from $L_1(E)$:

$$\{0\} \subset R_1 \cap S_1 \subset R_1 \subset R_1 + S_1 \subset E\ ;$$

$$\{0\} \subset R_1 \cap S_1 \subset S_1 \subset R_1 + S_1 \subset E\ .$$

We have one more property that we will use afterwards

Proposition 2.2.2 *The subspace $R_1 \cap S_1$ is a direct factor in S_1, i.e. there exists a M-invariant subspace A_1 (or $A(E)$) in S_1 such that $S_1 = (R_1 \cap S_1) \oplus A_1$.*

Notice that A_1 is a semisimple subrepresentation.
But A_1 is not "canonical" : it is not unique and it may happen that no such space is invariant under the automorphisms of M. So A_1 will be usefull to understand the full structure, but we will not keep it in our "canonical lattice.

We have $R_1 + S_1 = R_1 \oplus A_1$, and the quotient representations $(R_1 + S_1)/S_1$ and $S_1/R_1 \cap S_1$ are isomorphic to the subrepresentation A_1.

Definition 2.2.3 *A_1 is a direct factor in E, i.e. there exists a M-invariant subspace B_1 such $E = A_1 \oplus B_1$. Then $B_1 \supset R_1$, $(R_1 + S_1) \cap B_1 = R_1$ and $B_1 \cap S_1 = R_1 \cap S_1$.*

Once again, B_1 **is not canonical**.

2.2.3 Iteration

We will now iterate the preceding step. We describe below the second step.
We define $\widetilde{E_2} = R_1/(R_1 \cap S_1)$. It is a quotient representation of M. It may be a semisimple representation, in which case we stop the iteration. If not, we may apply the construction above (first step) to it.
We denote $\widetilde{R_2} = R(\widetilde{E_2})$ and $\widetilde{S_2} = S(\widetilde{E_2})$. We may also pick some $\widetilde{A_2} = A(\widetilde{E_2})$.
Then we go to $\widetilde{E_3} = \widetilde{R_2}/(\widetilde{R_2} \cap \widetilde{S_2})$.
We describe formally the iteration :
We begin with $\widetilde{E_1} = E$ and we define :

- $\widetilde{R_p} = R(\widetilde{E_p})$
- $\widetilde{S_p} = S(\widetilde{E_p})$
- $\widetilde{E_{p+1}} = \widetilde{R_p}/(\widetilde{R_p} \cap \widetilde{S_p})$

As long as the representation $\widetilde{E_p}$ is non zero and non semi-simple, then the subspaces $\widetilde{R_{p+1}}$ and $\widetilde{S_{p+1}}$ are non zero and different from $\widetilde{E_p}$, so the dimension of the quotient $\widetilde{E_{p+1}} = \widetilde{R_{p+1}}/(\widetilde{R_{p+1}} \cap \widetilde{S_{p+1}})$ is strictly less than the dimension of $\widetilde{E_p}$ (at least 2 dimension less). Since the dimension of E is finite, that iterative processe ends after a finite number of steps.
We stop it when $\widetilde{E_{p+1}}$ is a semisimple representation.

Definition 2.2.4 *The first p such that $\widetilde{E_{p+1}}$ is a semisimple representation is called the depth of the representation M. We will denote it by d.*

So for example, depth $d = 0$ means that M is semisimple.

2.2.4 First canonical lattice

Now the above spaces are subspaces of different quotients. More precisely for $p = 1$ we have $\widetilde{R_1} = R_1$ and $\widetilde{S_1} = S_1$ which are suspaces of E. But for $p > 1$ we get subspaces in quotients, and not subspaces of E itself. In order to get a lattice of subspaces in E we will "lift" all those spaces to E through the different quotients.
We define the quotients map $pr_p : \widetilde{R_p} \longrightarrow \widetilde{E_{p+1}}$ for $p = 1, \cdots, d$. Notice that $\widetilde{R_p} \subset \widetilde{E_p}$, so we may compose the pull-back $(pr_p)^{-1}$ in the definition below :

Definition 2.2.5 *For any $p = 2, \cdots, d$ we denote :*

- $R_p = (pr_1)^{-1} \circ \cdots \circ (pr_{p-1})^{-1}(\widetilde{R_p})$,
- $S_p = (pr_1)^{-1} \circ \cdots \circ (pr_{p-1})^{-1}(\widetilde{S_p})$.

We denote by $\mathcal{L}_\rfloor(\mathcal{M})$ the lattice of subspaces of E generated by the subspaces R_p and S_p for $p = 1, \cdots, d$ and we call it the first canonical lattice of M.

We recall that a lattice generated by a family of subspaces of E is the set of subspaces that one may get by successive applications of intersections and sums. By convention, that lattice contains $\{0\}$ and E.
Such a lattice may contains an infinite number of spaces, even if the given family is finte. But our canonical lattice has many properties.

Theorem 2.2.6 *The lattice $\mathcal{L}_\rfloor(\mathcal{M})$ is finite.*

More precisely, we may enumerate all the elements of $\mathcal{L}_\rfloor(\mathcal{M})$.
For any subset $V \subset \{1, \cdots, d\}$, we denote $S_V = S_{v_1} + S_{v_2} + \cdots + S_{v_m}$ where the v_j (with $1 \leqslant v_1 < v_2 < \cdots < v_m \leqslant d$) are the elements of V. For the case $V = \emptyset$, we set $S_\emptyset = \{0\}$. Similarly, we set by convention $R_0 = E$.

Proposition 2.2.7 *The subspaces $S(p, V) = (R_p \cap S_p) + S_V$ and $R(p, V) = R_p + S_V$, for all $p = 0, \cdots, d$ and all subsets $V \subset \{1, \cdots, p\}$, are all the elements of the lattice $\mathcal{L}_\rfloor(\mathcal{M})$.*

Some special cases are $R_p = R(p, \emptyset)$ and $S_p = S(p, \{p\})$.
We have also some inclusions :

- $(R_p \cap S_p) \subset (R_{p+1} \cap S_{p+1}) \quad \forall p = 1, \cdots, d-1$
- $R_{p+1} \subset R_p \quad \forall p = 1, \cdots, d-1$
- $S(p, V) \subset R(p, V)$
- $S(p, V) \subset S(p+1, W) \quad \forall V \subset W$
- $R(p+1, V) \subset R(p, W) \quad \forall V \subset W$

The first three inclusions are always strict, except perhaps the inclusion $S(d, V) \subset R(d, V)$.
On the other hand, the last two inclusions are not necessarily strict.
More precisely, at each step we may consider the subspace $\widetilde{A_p} = A(\widetilde{E_p})$ which is an invariant subspace of $\widetilde{E_p}$, supplementary of $\widetilde{R_p} \cap \widetilde{S_p}$ into $\widetilde{S_p}$. That space is semisimple and may be zero or not. We have $\widetilde{A_p} = \{0\}$ if and only if $\widetilde{S_p} \subset \widetilde{R_p}$.
With that, we see that the lattice $\mathcal{L}(\mathcal{M})$ has at most $2^{d+2} - 2$ elements [if all the $\widetilde{A_p}$ are non-zero, for all $p = 1, \cdots, d$] and at least $2d + 1$ elements [if all the $\widetilde{A_p}$ are zero].

2.2.5 Maximal flags

We will consider now some "maximal" flags in $\mathcal{L}(\mathcal{M})$, i.e. ordered families of subspaces in $\mathcal{L}(\mathcal{M})$: $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m \subset F_{m+1} = E$ such that for any $j = 1, \cdots, d$, we have $F_j \neq F_{j+1}$ and there is no other subspace of $\mathcal{L}(\mathcal{M})$ between F_j and F_{j+1}.
There may be many different maximal flags in $\mathcal{L}(\mathcal{M})$, but the "length" of those flags [i.e. the number $m+2$ of elements of the flag, including $\{0\}$ and E] depends only on $\mathcal{L}(\mathcal{M})$. That length depends on the number t_M of those p such that $\widetilde{A_p} = \{0\}$. We will describe some flags in the maximal case where $t_M = 0$, and it will be easy to see which spaces has to be suppressed in order to obtain a "true" maximal flag when $t_M \neq 0$.

Proposition 2.2.8 *The flag :*

- $F_i = S(i, \emptyset) = R_i \cap S_i \quad \text{for} 0 \leqslant i \leqslant d$
- $F_{d+j} = S(d, \{1, \cdots, j\}) = (R_d \cap S_d) + S_1 + \cdots + S_j \quad \text{for} 1 \leqslant j \leqslant d$
- $F_{2d+1} = R(d, \{1, \cdots, d\}) = R_d + S_1 + \cdots + S_d$
- $F_{2d+1+k} = R(d-k, \{1, \cdots, d-k\}) = R_{d-k} + S_1 + \cdots + S_{d-k} \quad \text{for} 1 \leqslant i \leqslant d$

is a flag in $\mathcal{L}(\mathcal{M})$, which is maximal (with length $3d+2$) when $t_M = 0$.
If $t_M \neq 0$, we suppress all the F_{d+j} such that $\widetilde{A_j} = \{0\}$ and we get a maximal flag in $\mathcal{L}(\mathcal{M})$ with length $3d + 2 - t_M$.

If all the $\widetilde{A_j} = \{0\}$, i.e. if $t_M = d+1$, then we suppress all the F_{d+j} forall$j = 1, \cdots, d+1$ above and the length of the maximal flag is $2d+1$.
For simplicity, we may consider the subflag given by suppressing all these F_{d+j} forall$j = 1, \cdots, d$. It is already sufficient for some applications that we will consider.

Theorem 2.2.9 *The flag :*

- $JH_i = F_i \quad \text{for} 0 \leqslant i \leqslant d$
- $JH_{d+1} = F_{2d+1}$
- $JH_{d+1+k} = F_{2d+1+k} \quad \text{for} 1 \leqslant i \leqslant d$

is a Jordan-Hölder decomposition of the representation M if $t_M \neq d$.
In the case $t_M = d$ [i.e. if we have $S_p \subset R_p$ $\forall p = 1, \cdots, d-1$ and $R_d = S_d$, then we have to suppress JH_{d+1} to obtain a Jordan-Hölder decomposition.

Notice that the proposition says that the quotient $(F_{2d+1})/(F_d) = (R_d + S_1 + \cdots + S_d)/(R_d \cap S_d)$ is indeed semisimple. It is isomorphic to the direct sum of all the representations $\widetilde{A_p}$ for all $p = 1, \cdots, d+1$.
For the successive (other) subquotients, which are always nonzero, we have

- For any $i = 1, \cdots, d, \quad JH_i/JH_{i-1}$ is isomorphic to $(\widetilde{R_i})/(\widetilde{R_i} \cap \widetilde{S_i})$

- For any $k = 1, \cdots, d, \quad JH_{d+1+k}/JH_{d+k}$ is isomorphic to $(\widetilde{E_{d+1-k}})/(\widetilde{R_{d+1-k}} + \widetilde{S_{d+1-k}})$

We may consider that Jordan-Hölder decomposition as the "minimal canonical" one.
Notice there other maximal flags in $\mathcal{L}(\mathcal{M})$ if $t_M \neq 0$. They depend on the places where we had the spaces S_p. Here are three of them :
Case 1 :

- $F_i^1 = F_i \quad$ for $i = 0, \cdots, d$ and $i = 2d+1, \cdots, 3d+2$

- $F_{d+j}^1 = R(d, \{d+2-j, \cdots, d\}) \quad$ for $j = 1, \cdots, d$

Case 2 :

- $F_{2i}^2 = S_1 + \cdots + S_i \quad$ for $i = 0, \cdots, d$

- $F_{2j-1}^2 = (R_j \cap S_j) + S_1 + \cdots + S_{j-1} \quad$ for $j = 1, \cdots, d$

- $F_{2d+1+k}^2 = F_{2d+1+k} \quad$ for $k = 0, \cdots, d$

Case 3 :

- $F_i^3 = F_i \quad$ for $i = 0, \cdots, d$

- $F_{d+2j}^3 = R_{d+1-j} + S_{d+1-j} \quad$ for $j = 1, \cdots, d$

- $F_{d+2k+1}^3 = R_{d-k} \quad$ for $k = 0, \cdots, d$

2.3 Additional structures

2.3.1 Complex and quaternionic representations

2.3.2 Reflexive structures

2.3.3 Orthogonality

2.3.4 Other structures

2.4 Adjoint representation

2.4.1

3 Semi-Riemannian symmetric spaces

3.1 titre de sous-section

3.1.1

4 Semi-simple case

4.1 titre de sous-section

4.1.1

5 Open questions

5.1 titre de sous-section

5.1.1